Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337

TMA 04 2020J

Covers Books A, B, C and D **Cut-off date 5 May 2021**

You can submit this TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on it, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' area of the M337 website.

Your work should be written in good mathematical style, as demonstrated by the example and exercise solutions in the study units. You should explain your solutions carefully, using appropriate notation and terminology, and write in sentences. As usual, you should simplify algebraic answers where possible.

In the wording of the questions:

- *write down* or *state* means 'write down without justification'
- *find*, *determine*, *calculate*, *explain*, *derive*, *evaluate* or *solve* means that we require you to show all your working in giving an answer
- *prove*, *show* or *deduce* means that you should carefully justify each step of your solution.

Make sure to reference any significant result from the module materials that you use, and check that all the conditions of the result are satisfied.

Question 1 (Unit D1) – 30 marks

(a) Let q be the velocity function

$$q(z) = \frac{i}{\overline{z}+1}.$$

(i) Explain why q is the velocity function for an ideal flow on $\mathbb{C} - \{-1\}$. [2]

(ii) Write down a complex potential function for q, and obtain the corresponding stream function. [3]

(iii) Determine equations involving a constant for the streamlines of q.

Hence find equations for the streamlines through the points 0, $-1+2i$ and $1-2i$, and sketch them on a single diagram, indicating the direction of flow in each case. [7]

(iv) Find the flux of q across the line segment Γ from $-1+i$ to $-1+2i$. [2]

(b) Consider the obstacle

$$K = [-3,-1] \cup \{z : |z| \le 1\} \cup [1,3].$$

(i) Use the result of Exercise 3.1(a) on page 38 of Book D to show that the Joukowski function J is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - [-10/3, 10/3]$. [3]

(ii) Use Theorem 3.2 on page 38 of Book D to show that the function

$$J_{5/3}(z) = z + \frac{25/9}{z}$$

is a one-to-one conformal mapping from $\{z : |z| > 5/3\}$ onto $\mathbb{C} - [-10/3, 10/3]$. [1]

(iii) Use the results of parts (b)(i) and (ii) to show that $f = J_{5/3}^{-1} \circ J$ is a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\{z : |z| > 5/3\}$. [2]

(iv) Use the Flow Mapping Theorem to deduce from part (b)(iii) that the solution to the Obstacle Problem for K with circulation 4π around K is

$$q(z) = \overline{\left(1 - \frac{2i}{J(z)\sqrt{1-(10/3J(z))^2}}\right) J'(z)}, \qquad [7]$$

by first finding a complex potential for this flow.

You may assume that the function f in part (b)(iii) satisfies the Laurent series condition of the Flow Mapping Theorem.

(v) Verify that $\lim_{z\to\infty} q(z) = 1$ for the function q in part (b)(iv). [3]

Question 2 (Unit D2) – 30 marks

(a) Let $f(z) = \frac{1}{2}z^2 + 3z + 5$.

(i) Find the fixed points α and β of the function f, and classify them as attracting, repelling or indifferent, identifying any attracting fixed points that are super-attracting. [5]

(ii) Prove that the iteration sequence

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \ldots,$$

is conjugate to the iteration sequence

$$w_{n+1} = P_{7/4}(w_n), \quad n = 0, 1, 2, \ldots,$$

and determine the conjugating function h. [3]

(iii) Verify that $h(\alpha)$ and $h(\beta)$ are the fixed points of $P_{7/4}$. [2]

(iv) Use Lemma 4.1 on page 143 of Book D to find a 2-cycle of $P_{7/4}$. Hence find a 2-cycle of f, and classify it as attracting (possibly super-attracting), repelling or indifferent. [4]

(v) Find a set of eight distinct points in the keep set $K_{7/4}$, and determine whether each of these points is an interior point or a boundary point of $K_{7/4}$. [5]

(vi) Determine whether or not $K_{7/4}$ is connected. [2]

(b) Determine which of the following points lie in the Mandelbrot set.

(i) $c = \frac{5}{6} + \frac{5}{6}i$ [3]

(ii) $c = -\frac{5}{6} + \frac{1}{6}i$ [3]

(iii) $c = \frac{1}{6} + \frac{1}{6}i$ [3]

The final four questions are practice examination-style questions covering material from Books A, B and C.

Question 3 – 10 marks

Determine each of the following complex numbers in *polar* form, simplifying your answers as far as possible.

(a) $-3 + \sqrt{3}\,i$ [2]

(b) $(-3 + \sqrt{3}\,i)^{-5}$ [2]

(c) $\dfrac{-\sqrt{3} + 3i}{-3 + \sqrt{3}\,i}$ [3]

(d) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{1+i}$ [3]

Question 4 – 10 marks

(a) Evaluate the following integrals in which Γ is the square contour with vertices 0, $1+i$, 2 and $1-i$. Name any standard results that you use, and check that their hypotheses are satisfied.

(i) $\displaystyle\int_\Gamma \frac{z-1}{z+1}\,dz$ [2]

(ii) $\displaystyle\int_\Gamma \frac{z+1}{z-1}\,dz$ [3]

(iii) $\displaystyle\int_\Gamma \left(\frac{z+1}{z-1}\right)^2 dz$ [4]

(b) Write down a square contour Γ for which exactly one of the three integrals from part (a) is non-zero. [1]

Question 5 – 10 marks

Let

$$f(z) = \sum_{n=1}^{\infty} n(1-z)^{n-1} \quad (|1-z| < 1)$$

and

$$g(z) = \sum_{n=1}^{\infty} n(z+1)^{n-1} \quad (|z+1| < 1).$$

(a) Explain why f and g are not direct analytic continuations of each other. [2]

(b) Use the binomial series

$$(1-w)^{-2} = \sum_{n=1}^{\infty} nw^{n-1}, \quad \text{for } |w| < 1,$$

to find an analytic function h such that

$$f(z) = h(z), \quad \text{for } |1-z| < 1,$$

and

$$g(z) = h(z), \quad \text{for } |z+1| < 1.$$ [5]

(c) Deduce that f and g are indirect analytic continuations of each other. [3]

Question 6 – 10 marks

(a) Find the image of the upper half-plane $H = \{z : \operatorname{Im} z > 0\}$ under the Möbius transformation

$$f(z) = \frac{z-i}{iz-1}.$$ [5]

(b) Find a one-to-one conformal mapping from the region

$$\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$$

onto the region

$$\mathcal{S} = \{z : |z-2i| < 1\}.$$ [5]